Limit theorem for a time-dependent coined quantum walk on the line

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Abstract. We study time-dependent discrete-time quantum walks on the one-dimensional lattice. We compute the limit distribution of a two-period quantum walk defined by two orthogonal matrices. For the symmetric case, the distribution is determined by one of two matrices. Moreover, limit theorems for two special cases are presented.

1 Introduction

The discrete-time quantum walk (QW) was first intensively studied by Ambainis *et al.* [1]. The QW is considered as a quantum generalization of the classical random walk. The random walker in position $x \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ at time $t\,(\in \{0,1,2,\dots\})$ moves to $x-1$ at time $t+1$ with probability p, or $x+1$ with probability $q\,(=1-p)$. In contrast, the evolution of the quantum walker is defined by replacing p and q with 2×2 matrices P and Q, respectively. Note that $U = P + Q$ is a unitary matrix. A main difference between the classical walk and the QW is seen on the particle spreading. Let $\sigma(t)$ be the standard deviation of the walk at time t. That is, $\sigma(t) = \sqrt{\mathbb{E}(X_t^2) - \mathbb{E}(X_t)^2}$, where X_t is the position of the quantum walker at time t and $\mathbb{E}(Y)$ denotes the expected value of Y. Then the classical case is a diffusive behavior, $\sigma(t) \sim \sqrt{t}$, while the quantum case is ballistic, $\sigma(t) \sim t$ (see [1], for example).
In the context of quantum computation, the QW is applied to several quantum algorithms. By using the quantum algorithm, we solve a problem quadratically faster than the corresponding classical algorithm. As a well-known quantum search algorithm, Grover's algorithm was presented. The algorithm solves the following problem: in a search space of N vertices, one can find a marked vertex. The corresponding classical search requires $O(N)$ queries. However, the search needs only $O(\sqrt{N})$ queries. As well as the Grover algorithm, the QW can also search a marked vertex with a quadratic speed up, see Shenvi *et al.* [2]. It has been reported that quantum walks on regular graphs (e.g., lattice, hypercube, complete graph) give faster searching than classical walks. The Grover search algorithm can also be interpreted as a QW on complete graph. Decoherence is an important concept in quantum information processing. In fact, decoherence on QWs has been extensively investigated, see Kendon [3], for example. However, we should note that our results

are not related to the decoherence in QWs. Physically, Oka *et al.* [4] pointed out that the Landau-Zener transition dynamics can be mapped to a QW and showed the localization of the wave functions.

In the present paper, we consider the QW whose dynamics is determined by a sequence of time-dependent matrices, $\{U_t : t = 0, 1, \dots\}$. Ribeiro *et al.* [5] numerically showed that periodic sequence is ballistic, random sequence is diffusive, and Fibonacci sequence is sub-ballistic. Mackay *et al.* [6] and Ribeiro *et al.* [5] investigated some random sequences and reported that the probability distribution of the QW converges to a binomial distribution by averaging over many trials by numerical simulations. Konno [7] proved their results by using a path counting method. By comparing with a position-dependent QW introduced by Wójcik *et al.* [8], Bañuls *et al.* [9] discussed a dynamical localization of the corresponding time-dependent QW.

In this paper, we present the weak limit theorem for the two-period time-dependent QW whose unitary matrix U_t is an orthogonal matrix. Our approach is based on the Fourier transform method introduced by Grimmett *et al.* [10]. We think that it would be difficult to calculate the limit distribution for the general n-period ($n = 3, 4, \dots$) walk. However, we find out a class of time-dependent QWs whose limit probability distributions result in that of the usual (i.e., one-period) QW. As for the position-dependent QW, a similar result can be found in Konno [11].

The present paper is organized as follows. In Sect. 2, we define the time-dependent QW. Section 3 treats the two-period time-dependent QW. By using the Fourier transform, we obtain the limit distribution. Finally, in Sect. 4, we consider two special cases of time-dependent QWs. We show that the limit distribution of the walk is the same as that of the usual one.

2 Time-dependent QW

In this section we define the time-dependent QWs. Let $|x\rangle$ ($x \in \mathbb{Z}$) be infinite components vector which denotes the position of the walker. Here, x-th component of $|x\rangle$ is 1 and the other is 0. Let $|\psi_t(x)\rangle \in \mathbb{C}^2$ be the amplitude of the walker in position x at time t, where $\mathbb{C}$ is the set of complex numbers. The time-dependent QW at time t is expressed by

$$|\Psi_t\rangle = \sum_{x\in\mathbb{Z}} |x\rangle \otimes |\psi_t(x)\rangle \,. \tag{1}$$

To define the time evolution of the walker, we introduce a unitary matrix

$$U_t = \begin{bmatrix} a_t & b_t \\ c_t & d_t \end{bmatrix}, \tag{2}$$

where $a_t, b_t, c_t, d_t \in \mathbb{C}$ and $a_t b_t c_t d_t \neq 0$ ($t = 0, 1, \dots$). Then U_t is divided into P_t and Q_t as follows:

$$P_t = \begin{bmatrix} a_t & b_t \\ 0 & 0 \end{bmatrix}, \; Q_t = \begin{bmatrix} 0 & 0 \\ c_t & d_t \end{bmatrix}. \tag{3}$$

The evolution is determined by

$$|\Psi_{t+1}\rangle = \sum_{x\in\mathbb{Z}} |x\rangle \otimes (P_t\,|\psi_t(x+1)\rangle + Q_t\,|\psi_t(x-1)\rangle)\,. \tag{4}$$

Let $||\,|y\rangle\,||^2 = \langle y|y\rangle$. The probability that the quantum walker X_t is in position x at time t, $P(X_t = x)$, is defined by

$$P(X_t = x) = ||\,|\psi_t(x)\rangle\,||^2. \tag{5}$$

Moreover, the Fourier transform $|\hat{\Psi}_t(k)\rangle$ ($k \in [0, 2\pi)$) is given by

$$|\hat{\Psi}_t(k)\rangle = \sum_{x\in\mathbb{Z}} e^{-ikx}\,|\psi_t(x)\rangle\,, \tag{6}$$

with $i = \sqrt{-1}$. By the inverse Fourier transform, we have

$$|\psi_t(x)\rangle = \int_0^{2\pi} \frac{dk}{2\pi} e^{ikx}\,|\hat{\Psi}_t(k)\rangle\,. \tag{7}$$

The time evolution of $|\hat{\Psi}_t(k)\rangle$ is

$$|\hat{\Psi}_{t+1}(k)\rangle = \hat{U}_t(k)\,|\hat{\Psi}_t(k)\rangle\,, \tag{8}$$

where $\hat{U}_t(k) = R(k)U_t$ and $R(k) = \begin{bmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{bmatrix}$. We should remark that $R(k)$ satisfies $R(k_1)R(k_2) = R(k_1 + k_2)$ and $R(k)^* = R(-k)$, where $*$ denotes the conjugate transposed operator. From (8), we see that

$$|\hat{\Psi}_t(k)\rangle = \hat{U}_{t-1}(k)\hat{U}_{t-2}(k)\cdots\hat{U}_0(k)\,|\hat{\Psi}_0(k)\rangle\,. \tag{9}$$

Note that, when $U_t = U$ for any t, the walk becomes a usual one-period walk, and $|\hat{\Psi}_t(k)\rangle = \hat{U}(k)^t\,|\hat{\Psi}_0(k)\rangle$. Then the probability distribution of the usual walk is

$$P(X_t = x) = \left|\left|\int_0^{2\pi} \frac{dk}{2\pi} e^{ikx}\hat{U}(k)^t\,|\hat{\Psi}_0(k)\rangle\right|\right|^2. \tag{10}$$

In Sect. 4, we will use this relation. In the present paper, we take the initial state as

$$|\psi_0(x)\rangle = \begin{cases} {}^T[\,\alpha,\,\beta\,]\ (x = 0) \\ {}^T[\,0,\,0\,]\ \ (x \neq 0) \end{cases}, \tag{11}$$

where $|\alpha|^2 + |\beta|^2 = 1$ and T is the transposed operator. We should note that $|\hat{\Psi}_0(k)\rangle = |\psi_0(0)\rangle$.

3 Two-period QW

In this section we consider the two-period QW and calculate the limit distribution. We assume that $\{U_t : t = 0, 1, \ldots\}$ is a sequence of orthogonal matrices with $U_{2s} = H_0$ and $U_{2s+1} = H_1$ $(s = 0, 1, \ldots)$, where

$$H_0 = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}, \; H_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}. \tag{12}$$

Let

$$f_K(x; a) = \frac{\sqrt{1-|a|^2}}{\pi(1-x^2)\sqrt{|a|^2 - x^2}}\, I_{(-|a|,|a|)}(x), \tag{13}$$

where $I_A(x) = 1$ if $x \in A$, $I_A(x) = 0$ if $x \notin A$. Then we obtain the following main result of this paper:

Theorem 1.

$$\frac{X_t}{t} \Rightarrow Z, \tag{14}$$

where $\Rightarrow$ means the weak convergence (i.e., the convergence of the distribution) and Z has the density function $f(x)$ as follows:
(i) If $\det(H_1 H_0) > 0$, then

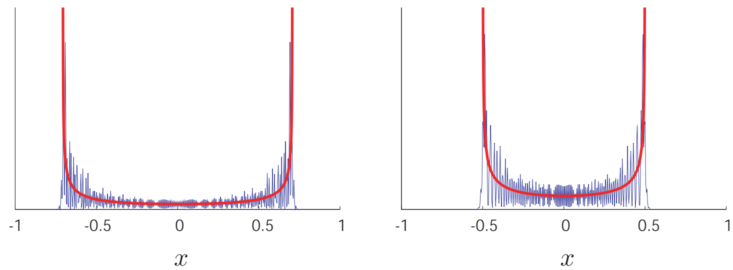
$$f(x) = f_K(x; a_\xi)\left[1 - \left\{|\alpha|^2 - |\beta|^2 + \frac{\left(\alpha\overline{\beta} + \overline{\alpha}\beta\right) b_0}{a_0}\right\} x\right], \tag{15}$$

where $|a_\xi| = \min\{|a_0|, |a_1|\}$.

(ii) If $\det(H_1 H_0) < 0$, then

$$f(x) = f_K(x; a_0 a_1)\left[1 - \left\{|\alpha|^2 - |\beta|^2 + \frac{\left(\alpha\overline{\beta} + \overline{\alpha}\beta\right) b_0}{a_0}\right\} x\right]. \tag{16}$$

If the two-period walk with $\det(H_1 H_0) > 0$ has a symmetric distribution, then the density of Z becomes $f_K(x; a_\xi)$. That is, Z is determined by either H_0 or H_1. Figure 1 (a) shows that the limit density of the two-period QW for $a_0 = \cos(\pi/4)$ and $a_1 = \cos(\pi/6)$ is the same as that for the usual (one-period) QW for a_0, since $|a_0| < |a_1|$. Similarly, Fig. 1 (b) shows that the limit density of the two-period QW for $a_0 = \cos(\pi/4)$ and $a_1 = \cos(\pi/3)$ is equivalent to that for the usual (one-period) QW for a_1, since $|a_0| > |a_1|$.



(a) $a_0 = \cos(\pi/4)$, $a_1 = \cos(\pi/6)$

(b) $a_0 = \cos(\pi/4)$, $a_1 = \cos(\pi/3)$

Fig. 1. The limit density function $f(x)$ (thick line) and the probability distribution at time $t = 500$ (thin line).

Proof. Our approach is due to Grimmett *et al.* [10]. The Fourier transform becomes

$$|\hat{\Psi}_{2t}(k)\rangle = \left(\hat{H}_1(k)\hat{H}_0(k)\right)^t |\hat{\Psi}_0(k)\rangle \,, \tag{17}$$

where $\hat{H}_\gamma(k) = R(k)H_\gamma \; (\gamma = 0,1)$. We assume

$$H_\gamma = \begin{bmatrix} \cos\theta_\gamma & \sin\theta_\gamma \\ \sin\theta_\gamma & -\cos\theta_\gamma \end{bmatrix}, \tag{18}$$

with $\theta_\gamma \neq \frac{\pi n}{2} \, (n \in \mathbb{Z})$ and $\theta_0 \neq \theta_1$. For the other case, the argument is nearly identical to this case, so we will omit it. The two eigenvalues $\lambda_j(k) \, (j = 0,1)$ of $\hat{H}_1(k)\hat{H}_0(k)$ are given by

$$\lambda_j(k) = c_1c_2\cos 2k + s_1s_2 + (-1)^j \, i\sqrt{1 - (c_1c_2\cos 2k + s_1s_2)^2}, \tag{19}$$

where $c_\gamma = \cos\theta_\gamma$, $s_\gamma = \sin\theta_\gamma$. The eigenvector $|v_j(k)\rangle$ corresponding to $\lambda_j(k)$ is

$$|v_j(k)\rangle = \begin{bmatrix} s_1c_2e^{2ik} - c_1s_2 \\ \left\{-c_1c_2\sin 2k + (-1)^j\sqrt{1-(c_1c_2\cos 2k + s_1s_2)^2}\right\} i \end{bmatrix}. \tag{20}$$

The Fourier transform $|\hat{\Psi}_0(k)\rangle$ is expressed by normalized eigenvectors $|v_j(k)\rangle$ as follows:

$$|\hat{\Psi}_0(k)\rangle = \sum_{j=0}^{1} \langle v_j(k)|\hat{\Psi}_0(k)\rangle \, |v_j(k)\rangle \,. \tag{21}$$

Therefore we have

$$\begin{aligned} |\hat{\Psi}_{2t}(k)\rangle &= \left(\hat{H}_1(k)\hat{H}_0(k)\right)^t |\hat{\Psi}_0(k)\rangle \\ &= \sum_{j=0}^{1} \lambda_j(k)^t \, \langle v_j(k)|\hat{\Psi}_0(k)\rangle \, |v_j(k)\rangle \,. \end{aligned} \tag{22}$$

The r-th moment of X_{2t} is

$$
\begin{aligned}
E((X_{2t})^r) &= \sum_{x\in\mathbb{Z}} x^r P(X_{2t}=x) \\
&= \int_0^{2\pi} \frac{dk}{2\pi} \langle \hat{\Psi}_{2t}(k)| \left(D^r |\hat{\Psi}_{2t}(k)\rangle \right) \\
&= \int_0^{2\pi} \sum_{j=0}^{1} (t)_r \lambda_j(k)^{-r} (D\lambda_j(k))^r \left| \langle v_j(k)|\hat{\Psi}_0(k)\rangle \right|^2 \\
&\quad + O(t^{r-1}),
\end{aligned} \tag{23}
$$

where $D = i(d/dk)$ and $(t)_r = t(t-1)\times\cdots\times(t-r+1)$. Let $h_j(k) = D\lambda_j(k)/2\lambda_j(k)$. Then we obtain

$$
E((X_{2t}/2t)^r) \to \int_{\Omega_0} \frac{dk}{2\pi} \sum_{j=0}^{1} h_j^r(k) |\, \langle v_j(k)|\hat{\Psi}_0(k)\rangle\, |^2 \quad (\, t\to\infty). \tag{24}
$$

Substituting $h_j(k) = x$, we have

$$
\lim_{t\to\infty} E((X_{2t}/2t)^r) = \int_{-|c_\xi|}^{|c_\xi|} x^r\, f(x)\, dx, \tag{25}
$$

where

$$
f(x) = f_K(x; c_\xi) \left[1 - \left\{ |\alpha|^2 - |\beta|^2 + \frac{(\alpha\bar{\beta} + \bar{\alpha}\beta)s_1}{c_1} \right\} x \right], \tag{26}
$$

and $|c_\xi| = |\cos\theta_\xi| = \min\{|\cos\theta_0|, |\cos\theta_1|\}$. Since $f(x)$ is the limit density function, the proof is complete. $\square$

4 Special cases in time-dependent QWs

In the previous section, we have obtained the limit theorem for the two-period QW determined by two orthogonal matrices. For other two-period case and general n-period ($n \geq 3$) case, we think that it would be hard to get the limit theorem in a similar fashion. Here we consider two special cases in the time-dependent QWs and give the weak limit theorems.

4.1 Case 1

Let us consider the QW whose evolution is determined by the following unitary matrix:

$$
U_t = \begin{bmatrix} ae^{iw_t} & b \\ c & de^{-iw_t} \end{bmatrix}, \tag{27}
$$

with $a,b,c,d \in \mathbb{C}$. Here $w_t \in \mathbb{R}$ satisfies $w_{t+1} + w_t = \kappa_1$, where $\kappa_1 \in \mathbb{R}$ and $\mathbb{R}$ is the set of real numbers. Note that κ_1 does not depend on time. In this case, w_t can be written as $w_t = (-1)^t(w_0 - \frac{\kappa_1}{2}) + \frac{\kappa_1}{2}$. Therefore the period of the QW becomes two. We should remark that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $(\equiv U)$ is a unitary matrix. Then we have

Theorem 2.

$$\frac{X_t}{t} \Rightarrow Z_1, \tag{28}$$

where Z_1 has the density function $f_1(x)$ as follows:

$$f_1(x) = f_K(x;a)\left\{1 - \left(|\alpha|^2 - |\beta|^2 + \frac{a\alpha\overline{b\beta}e^{iw_0} + \overline{a\alpha}b\beta e^{-iw_0}}{|a|^2}\right)x\right\}. \tag{29}$$

Proof. The essential point of this proof is that this case results in the usual walk. First we see that U_t can be rewritten as

$$\begin{aligned} U_t &= \begin{bmatrix} e^{iw_t/2} & 0 \\ 0 & e^{-iw_t/2} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{iw_t/2} & 0 \\ 0 & e^{-iw_t/2} \end{bmatrix} \\ &= R\left(\frac{w_t}{2}\right) U R\left(\frac{w_t}{2}\right). \end{aligned} \tag{30}$$

From this, the Fourier transform $|\hat{\Psi}_t(k)\rangle$ can be computed in the following.

$$\begin{aligned} |\hat{\Psi}_t(k)\rangle &= \left\{R(k)R\left(\frac{w_{t-1}}{2}\right) U R\left(\frac{w_{t-1}}{2}\right)\right\} \left\{R(k)R\left(\frac{w_{t-2}}{2}\right) U R\left(\frac{w_{t-2}}{2}\right)\right\} \\ &\quad \cdots \left\{R(k)R\left(\frac{w_0}{2}\right) U R\left(\frac{w_0}{2}\right)\right\} |\hat{\Psi}_0(k)\rangle \\ &= R\left(-\frac{w_t}{2}\right) \left\{R\left(\frac{w_t}{2}\right) R(k) R\left(\frac{w_{t-1}}{2}\right) U\right\} \\ &\quad \times \left\{R\left(\frac{w_{t-1}}{2}\right) R(k) R\left(\frac{w_{t-2}}{2}\right) U\right\} \\ &\quad \times \cdots \times \left\{R\left(\frac{w_1}{2}\right) R(k) R\left(\frac{w_0}{2}\right) U\right\} R\left(\frac{w_0}{2}\right) |\hat{\Psi}_0(k)\rangle \\ &= R\left(-\frac{w_t}{2}\right) \left\{R(k+\kappa_1/2)U\right\}^t R\left(\frac{w_0}{2}\right) |\hat{\Psi}_0(k)\rangle. \end{aligned} \tag{31}$$

Therefore we have

$$\begin{aligned} |\psi_t(x)\rangle &= \int_0^{2\pi} \frac{dk}{2\pi} e^{ikx} |\hat{\Psi}_t(k)\rangle = \int_{\kappa_1/2}^{2\pi+\kappa_1/2} \frac{dk}{2\pi} e^{i(k-\kappa_1/2)x} |\hat{\Psi}_t(k-\kappa_1/2)\rangle \\ &= e^{-i\kappa_1 x/2} R\left(-\frac{w_t}{2}\right) \int_{\kappa_1/2}^{2\pi+\kappa_1/2} \frac{dk}{2\pi} e^{ikx} \left(R(k)U\right)^t |\hat{\Psi}_0^R(k)\rangle, \end{aligned} \tag{32}$$

where $|\hat{\Psi}_0^R(k)\rangle = R\left(\frac{w_0}{2}\right) |\hat{\Psi}_0(k-\kappa_1/2)\rangle$. Then the probability distribution is

$$\begin{aligned} &P(X_t = x) \\ &= \left\{ e^{i\kappa_1 x/2} \left(\int_{\kappa_1/2}^{2\pi+\kappa_1/2} \frac{dk}{2\pi} e^{ikx} \left(R(k)U\right)^t |\hat{\Psi}_0^R(k)\rangle \right)^* R\left(\frac{w_t}{2}\right) \right\} \\ &\quad \times \left\{ e^{-i\kappa_1 x/2} R\left(-\frac{w_t}{2}\right) \left(\int_{\kappa_1/2}^{2\pi+\kappa_1/2} \frac{dk}{2\pi} e^{ikx} \left(R(k)U\right)^t |\hat{\Psi}_0^R(k)\rangle \right) \right\} \\ &= \left\| \int_{\kappa_1/2}^{2\pi+\kappa_1/2} \frac{dk}{2\pi} e^{ikx} \hat{U}(k)^t |\hat{\Psi}_0^R(k)\rangle \right\|^2, \end{aligned} \tag{33}$$

where $\hat{U}(k) = R(k)U$. This implies that Case 1 can be considered as the usual QW with the initial state $|\hat{\Psi}_0^R(k)\rangle = R\left(\frac{w_0}{2}\right)|\hat{\Psi}_0(k-\kappa_1/2)\rangle$ and the unitary matrix U. Then the initial state becomes

$$|\hat{\Psi}_0^R(k)\rangle = {}^T[e^{iw_0/2}\alpha,\, e^{-iw_0/2}\beta], \tag{34}$$

that is,

$$|\psi_0(x)\rangle = \begin{cases} {}^T[e^{iw_0/2}\alpha,\, e^{-iw_0/2}\beta] & (x=0) \\ {}^T[0,\, 0] & (x \neq 0) \end{cases}. \tag{35}$$

Finally, by using the result in Konno [12, 13], we can obtain the desired limit distribution of this case. □

4.2 Case 2

Next we consider the QW whose dynamics is defined by the following unitary matrix:

$$U_t = \begin{bmatrix} a & be^{iw_t} \\ ce^{-iw_t} & d \end{bmatrix}. \tag{36}$$

Here $w_t \in \mathbb{R}$ satisfies $w_{t+1} = w_t + \kappa_2$, where $\kappa_2 \in \mathbb{R}$ does not depend on t. In this case, w_t can be expressed as $w_t = \kappa_2 t + w_0$. Noting $U_t = R\left(\frac{w_t}{2}\right)UR\left(-\frac{w_t}{2}\right)$, we get a similar weak limit theorem as Case 1:

Theorem 3.

$$\frac{X_t}{t} \Rightarrow Z_2, \tag{37}$$

where Z_2 has the density function $f_2(x)$ as follows:

$$f_2(x) = f_K(x;a)\left\{1 - \left(|\alpha|^2 - |\beta|^2 + \frac{a\alpha\overline{b\beta}e^{-iw_0} + \overline{a\alpha}b\beta e^{iw_0}}{|a|^2}\right)x\right\}. \tag{38}$$

If $w_t = 2\pi t/n\,(n = 1, 2, \ldots)$, $\{U_t\}$ becomes an n-period sequence. In particular, when $n = 2$ and $a, b, c, d \in \mathbb{R}$, $\{U_t\}$ is a sequence of two-period orthogonal matrices. Then Theorem 3 is equivalent to Theorem 1 (i).

5 Conclusion and Discussion

In the final section, we draw the conclusion and discuss our two-period walks. The main result of this paper (Theorem 1) implies that if $\det(H_1H_0) > 0$ and $\min\{|a_0|, |a_1|\} = |a_0|$, then the limit distribution of the two-period walk is determined by H_0. On the other hand, if $\det(H_1H_0) > 0$ and $\min\{|a_0|, |a_1|\} = |a_1|$, or $\det(H_1H_0) < 0$, then the limit distribution is determined by both H_0 and H_1.

Here we discuss a physical meaning of our model. We should remark that the time-dependent two-period QW is equivalent to a position-dependent

two-period QW if and only if the probability amplitude of the odd position in the initial state is zero. In quantum mechanics, the Kronig-Penney model, whose potential on a lattice is periodic, has been extensively investigated, see Kittel [14]. A derivation from the discrete-time QW to the continuous-time QW, which is related to the Schrödinger equation, can be obtained by Strauch [15]. Therefore, one of interesting future problems is to clarify a relation between our discrete-time two-period QW and the Kronig-Penney model.

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